Experimental determination and validation of the bending and torsional stiffness of an alpine ski

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***Abstract*—During the development of an alpine ski, several prototypes are built until the desired ski performance is achieved by the analogous stiffness distribution over the ski length. So far, there is no technology to quantify the stiffness behaviour of an alpine ski, only qualitative methods. An alpine ski consists of several layers with different E-modules, cross sections, and lengths. As a result, the bending and torsional stiffness changes continuously along the length of the ski. However, the bending line and the torsion angle changing over the latitude can be measured under a defined load. With these three quantities, the bending and torsional stiffness distribution can be calculated over the length of an alpine ski. A test bench is developed to measure the desired quantities under laboratory conditions. A validation of the measurement for bending and torsional stiffness is then performed. The results can be used for comparison with previously developed skis and in future to allow conclusions to be drawn about the specific behaviour of the materials which are used. Also, for quality management, if the stiffness distribution is known, every ski of the same series should have the same stiffness distribution.**

I. Introduction

An alpine ski can be subjected into two different load condition during driving. The first is bending when the ski is going straight over a small hill. The other is torsion when the ski enters a turn and is loaded on the edge. An alpine ski has a complex shape and in follow constantly changes its cross-section over its length. The material distribution also changes constantly along its length. As a result, the stiffness is constantly changing over its length. The research question is whether it is possible to measure the stiffness distribution of a ski in both bending and torsional states as a function of twist as well as the bending line under load. Two previous master's theses have already been carried out. The first approach was to measure the distance between a reference position and the current position of the ski under load. Then, two derivations must be performed to obtain the curvature and solve the inverse problem of the differential equation of the bending line. The same procedure is performed for the torsional load condition. For each derivative performed, the uncertainty increases significantly [1]. In the second thesis, the same mechanical measurement approach is used, but the measurement points are interpolated along the length. An optimization algorithm is developed to interpolate the measurement along the length of the ski to minimize the error. In addition, a regulation is implemented to avoid oscillation of the optimization algorithm, which leads to an enormous computational effort. However, an acceptable result was obtained for the bending stiffness [2]. The new approach is to measure the tangential angular change of the bending line as the torsional twist along the length of the ski under load. As a result, only one derivative needs to be performed. Furthermore, a direct method for solving the inverse problem of the differential equation of torsion and bending can be performed. The measurement and validation of bending stiffness is already sufficiently defined via the Bernoulli Beam theory [2]. Torsional stiffness has the problem that an ideal symmetrical body is assumed. However, each ski has a different preload in the middle, which leads to a shifted rotation of the body axis to the centre of gravity axis [4]. First, the theoretical background is explained. Then the procedure for the measurement is explained. In the next step, the verification is performed. Finally, the results are discussed.

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II. Method

In section II-A the mathematical foundation for the evaluation is given. Also the evaluation of the axis shift at torsion is considered. In addition, the method for evaluating the measurement results via the differential equations is described. In section II-B, the mechanical part and procedure of the measurement is explained. Next, in section II-C the steps for preparing the raw data are described. Finally, the methods for verifying the accuracy are explained.

A. Math

For the evaluation of the bending stiffness, the Euler-Bernulli Theory is used. According to [2], the assumption of a shear-rigid bending is sufficient. Equation 1 shows the differential equation of the bending line. The curvature can be expressed as $\frac{\partial^2 w}{\partial x^2} = w_{,xx}(x)$ and the multiplication $EI(x) = E(x)I(x)$ which embodies the bending stiffness. The mechanical model can be seen in figure 1, from which the boundary conditions can be derived [3]. Already the first derivative of $w_{,x}(x)$ by $\phi(x)$ is given by the measurement procedure.

$$\phi_{,x}(x) = -\frac{M_y(x)}{EI(x)} \tag{1}$$

Then the moment distribution can be described in the equation 2 as follows.

$$M_y(x) = \begin{cases} F_{Flex}\,\frac{l}{L}\,x, & 0 \le x \le l \\ F_{Flex}\,\frac{l}{L}\,(L-x), & l \le x \le L \end{cases} \tag{2}$$

To determine the bending stiffness, the differential equation must be used as in equation 3.

$$EI(x) = -\frac{M_y(x)}{\phi_{,x}(x)} \tag{3}$$

The same analogy can be applied to torsion. However, by the simplified assumption of [4] only shear stresses due to the shear modulus $G(x)$ are considered. Due to the axial displacement depending on the pretension of a ski, it is expressed in [5] that small displacements have no influence on the torsion measurement. The differential equation of torsion can be expressed by the equation 4. The first derivative of the torsional twist can be expressed as $\frac{\partial\theta}{\partial x} = \theta_{,x}(x)$ with torsional stiffness $GI(x) = G(x)I(x)$.

$$\theta_{,x}(x) = \frac{M_t(x)}{GI(x)} \tag{4}$$

To determine the torsional stiffness, the differential equation can be substituted into the equation 5.

$$GI(x) = \frac{M_t(x)}{\theta_{,x}(x)} \tag{5}$$

Since the applied force for both bending and torsion is known at all times via a load cell in the test bench and the angular change ϕ and θ can be measured, the stiffness can be determined directly via this analogy.

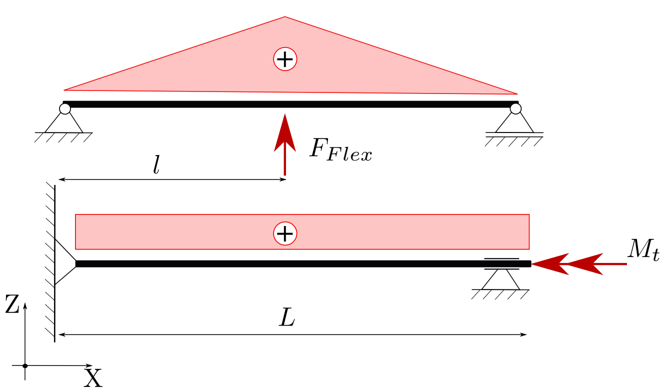

Fig. 1. Mechanical model of for bending and torsion with corresponding moment distribution

B. Procedure

In the figure 2 can be seen the overall structure of the test bench. The test bench can measure the bending line and the torsion twist along the ski in two separate measuring passes via the Lifting device and a Flexible Clamp.

For the bending measurement, the two clamps at the tips of the ski stay in neutral position. The Lifting device

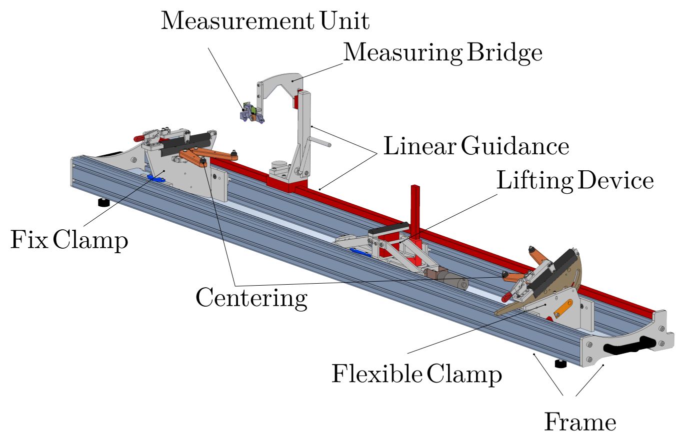

Fig. 2. Overall view of the test bench

in the middle apply a force to bend the ski. For the torsion mode, the Lifting device stay in rest position. The Flexible Clamp can be rotated up to 30° around the intersection of the plane of symmetry and the base of the ski. The applied torque can be measured via a load cell inside the Flexible Clamp. The Measurement Unit in figure 3 has two rockers, each with one degree of freedom about the X-axis and the Y-axis. Two incremental encoders are used to measure the change in angle while sliding over the ski base surface. The sensitivity of the encoders are 0.01°.

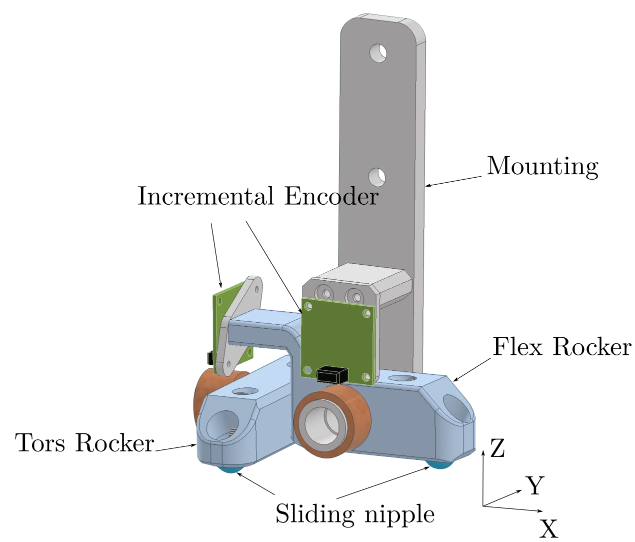

Fig. 3. The seesaws with encoder for bending and torsional measurement

A simplified three-point bending test can be adopted for the bending. In the center at the Lifting device, the applied force for the deflection is measured. All distances are known. The Measurement Unit is mounted on a carriage with free vertical guidance. The horizontal position change is recorded via an incremental encoder. During torsion, the flexible clamp is rotated by a maximum of 30°. The pivot point is located on the surface of the ski base. The measurement is made in the same way as before, only this time with the torsion seesaw. The force applied for the twist is measured by a load cell inside the Flexible Clamp. For data analysis, the measurement data is recorded by a Teensy4.1 microcontroller. The recorded data is sent to a local computer via a serial connection.

C. Evaluation procedure

For the application of the direct method to solve the inverse problem of the differential equation in bending and torsion, the received data must be prepared. First, all data points are deleted where no change from the previous data point occurs. Second, the data points are smoothed by a weighted moving average called Loess algorithm [6]. In the third step, the data points are interpolated by a cubic spline. Within the continuous function values $f(x)$ of the cubic spline, a symmetric derivative is applied according to the equation 6 demonstrated for bending.

$$\phi_{,x} = \frac{f(x+h) - f(x-h)}{2h} \tag{6}$$

The step size h is related to the sensitivity factor of the position encoder in the X-direction, which is given by $0.01884\,\mathrm{mm}$.

D. Verification

To check the test setup, a sheet metal strip with a known constant cross-section and modulus of elasticity is clamped in the test bench and measured. Since the shape of the curve is known, the difference can be determined from the measurement data which is generated. This procedure can be performed with several sheet metal strips and different contours, which are shown in figure 4. To check accuracy, the measurement is then performed $N = 8$ times with an alpine ski under constant load conditions. At each X-position, the range of standard deviation can be determined. Finally, the stiffness results of two identical skis can be compared.

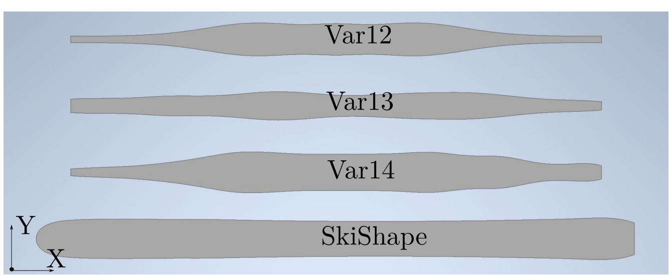

Fig. 4. Four metal sheets with constant thickness and varying width

To check the measured contour for correctness, a milled profile corresponding to the polynomial function $F(x) = 0.001x^2 - 0.0000001x^4$ is also traversed with the Measurement Unit. In this case, the measurement is performed only through the bending rocker. The measurement setup can be seen in Figure 5. The measurement data must fit the first derivative of the function $F(x)$, which is $f(x) = 0.002x \cdot 0.0000004x^3$.

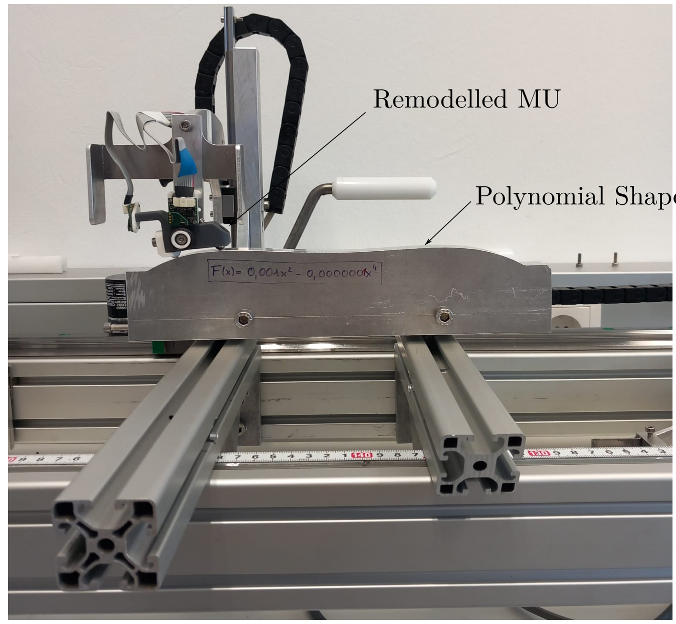

Fig. 5. A special configuration of the measurement unit with only one bending rocker for the polynomial function validation

III. Results

A. Accuracy

For accuracy, the measurement is performed $N = 8$ times. The standard deviation is determined from the raw measurement data ϕ and θ. Figure 6 shows the standard deviation of the bending at each X-position.

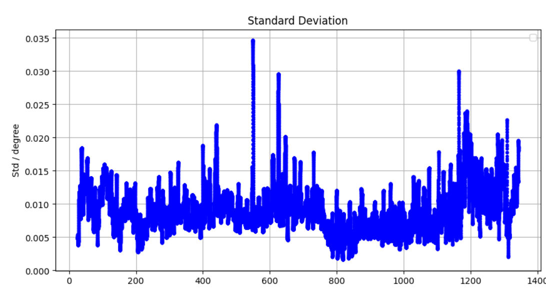

Fig. 6. Standard deviation of a bending measurement set with $N = 8$ measurements

The standard deviation can be given as $\overline{\sigma}_{w_{Flex}} = 0.0091°$ for bending. The mean standard deviation for torsion can be seen in Figure 7.

The mean standard deviation for torsion can be given as $\overline{\sigma}_{w_{Tors}} = 0.0124°$.

B. Polynomial function validation

For the polynomial function validation, a measurement is performed. In the figure 8 the graph of the measurement and the polynomial function $f(x)$ are shown together.

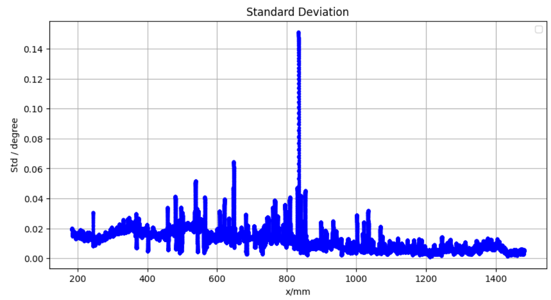

Fig. 7. Standard deviation of a torsion measurement set with $N = 8$ measurements

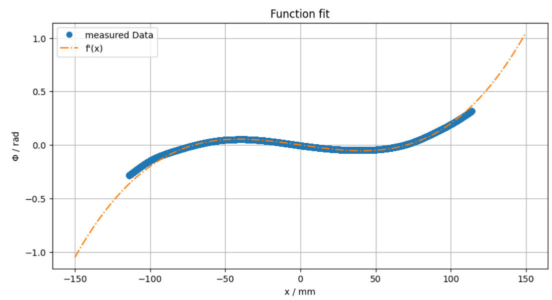

Fig. 8. Fit of the measurement to the analytic solution of the polynomial function

For the first derivative, the polynomial function is derived analytically and the measurement data via a symmetric derivative. Figure 9 shows the derived data.

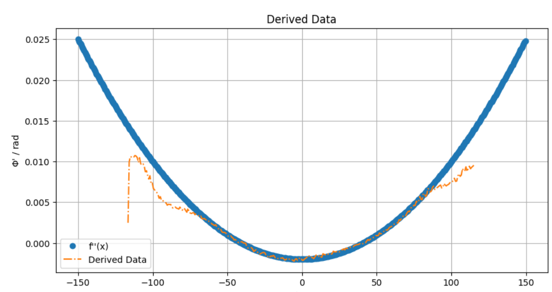

Fig. 9. First derivative of the measurement and the analytic solution

Significant discrepancies occur at the beginning and end of the measurement data. The maximum difference can be specified as $0.58°$. In the range of $-70 \leq x \leq 70$ the mean difference can be specified as $0.24°$.

C. *Constant Stiffness validation*

For the primary stiffness validation, a metal sheet with a known constant cross-section is measured in bending and torsion mode. Figure 12 shows the evaluated, calculated and averaged bending stiffness.

The averaged and calculated stiffness are relatively close to each other with a difference of about 1.79 %. The evaluated stiffness oscillates around the calculated stiffness,

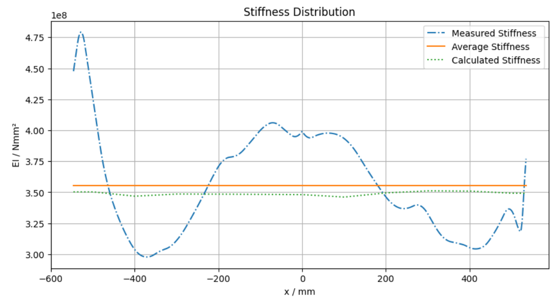

Fig. 10. Evaluated, calculated and average stiffness of a speciem with constant stiffness

but with larger deviations. Figure 11 shows the absolute stiffness difference of the bending measurement.

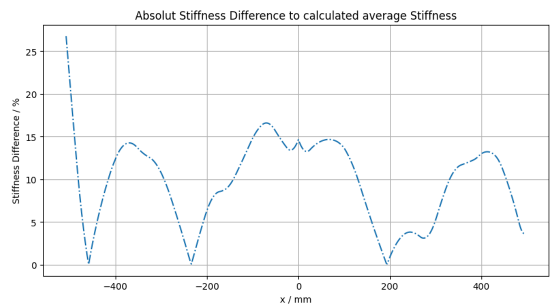

Fig. 11. Difference of the bending stiffness to the calculated stiffness

The peak difference can be given as 26.77 % and a mean difference of 9.72 % in the range of $-400 \leq x \leq 400$. Outside this range, the drift of the curve is much larger. Figure 12 shows the evaluated torsional stiffness with the averaged torsional stiffness.

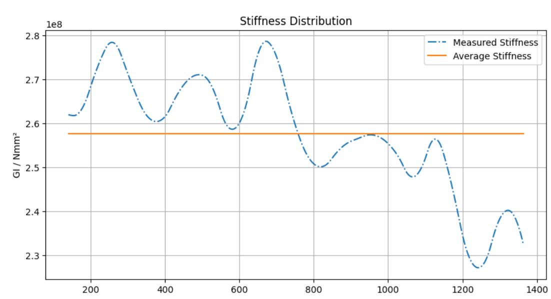

Fig. 12. Evaluated and averaged torsional stiffness of a specimen with constant stiffness

The same behaviour as before, the curve oscillates around the averaged stiffness. At the end, the graph also starts to develop larger drifts. Figure 13 shows the absolute difference between averaged and evaluated stiffness.

A peak difference of 11.9 % with a mean difference of 4.2 % in a range of $230 \leq x \leq 1230$ can be achieved.

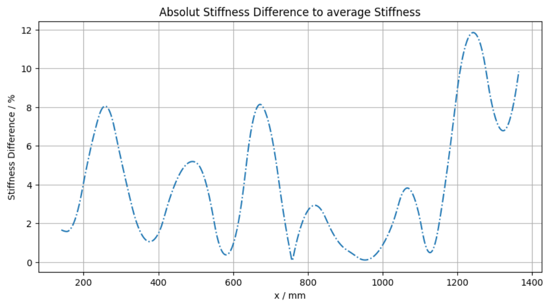

Fig. 13. Difference of the evaluated stiffness to the averaged stiffness

D. Variable Stiffness distribution

The results of a plate of profile shape $Var12$ are shown in bending and torsion. The figure 14 shows the evaluated, calculated and averaged bending stiffness.

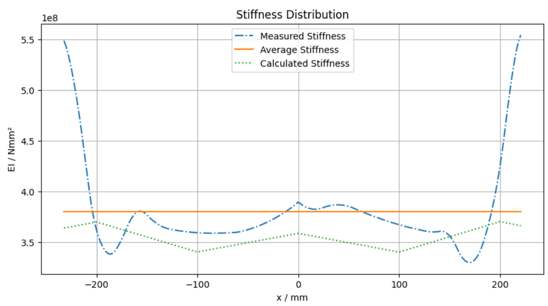

Fig. 14. Evaluated, averaged and calculated bending stiffness of the profile $Var12$

A similar curve progression can be observed in the range $-200 \leq x \leq 200$ from calculated to evaluated curve. The peak difference in this range can be given as 10.3 % with a mean difference of 5.8 %. At the ends, the evaluated curve starts to drift away from the calculated stiffness again. In figure 15 the same analogy with bending can be observed.

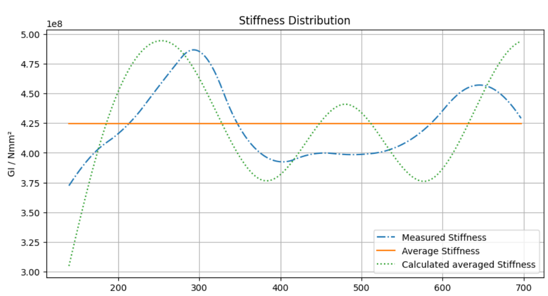

Fig. 15. Evaluated, averaged and calculated torsional stiffness of the profile $Var12$

The end also begins to drift, but with less intensity. The peak difference can be given as 22.2 % by a mean difference of 6.65 %.

E. Alpine Ski

Two identical skis are measured independently in bending and torsion. The stiffness distribution of the skis is not known. Figure 16 shows the evaluated bending stiffness of the two skis and the local mean value.

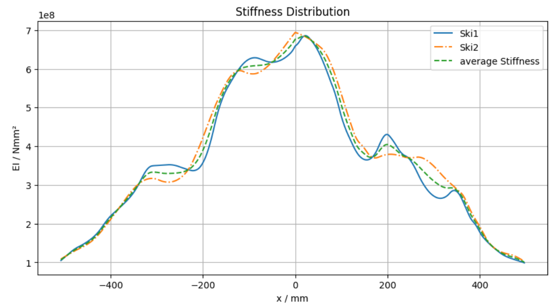

Fig. 16. Bending stiffness of two identical skis

Unlike the previous measurements, the stiffness distribution and the ski tips are close together. In the middle, a peak difference from the averaged stiffness can be seen at 12.5 % with a mean difference of 3.38 %. Figure 17 shows the torsional stiffness distribution.

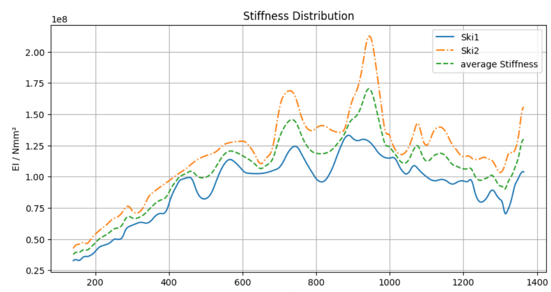

Fig. 17. Torsion stiffness of two identical skis

As before, the stiffnesses at the tips are close to each other. A peak difference of 24.99 % with a mean difference of 12.26 % can be observed.

IV. Conclusions

It can be said that the approach with the sliding measurement unit and the direct evaluation of the inverse problem of the differential equation works. However, the evaluation procedure cannot provide a sufficiently accurate result to make a clear statement about the stiffness distribution of an alpine ski. At least a peak difference of less than 8 % and a mean difference of about 4 % must be achieved continuously for several measurements [5]. A possible approach could be an optimization algorithm used in the evaluation to minimize the error between theoretical stiffness and measured stiffness.

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